

Lorentz invariant and supersymmetric interpretation of noncommutative quantum field theory

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Abstract

In this paper, using a Hopf-algebraic method, we construct deformed Poincaré SUSY algebra in terms of twisted (Hopf) algebra. By adapting this twist deformed super-Poincaré algebra as our fundamental symmetry, we can see the consistency between the algebra and non(anti)commutative relation among (super)coordinates and interpret that symmetry of non(anti)commutative QFT is in fact twisted one. The key point is validity of our new twist element that guarantees non(anti)commutativity of space. It is checked in this paper for $\mathcal{N} = 1$ case. We also comment on the possibility of noncommutative central charge coordinate. Finally, because our twist operation does not break the original algebra, we can claim that (twisted) SUSY is not broken in contrast to the string inspired $\mathcal{N} = 1/2$ SUSY in $\mathcal{N} = 1$ non(anti)commutative superspace.

1 Introduction

Quantum field theory on the noncommutative space-time has been under intensive study in recent years. This idea originates from Snyder's work [1]. They considered space-time noncommutativity can smear the UV divergence and give naive cut off of the theory. Stringy origin of

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noncommutativity of space-time is realized by considering the string propagating on *constant* NS-NS B-field background [2]. In that situation, we can find the space coordinates on D-brane become noncommutative, *i.e.*

$$[x^i, x^j] = i\Theta^{ij} \neq 0, \quad (1)$$

by taking appropriate zero-slope limit. Here, noncommutative parameter Θ^{ij} naively corresponds to VEV of background field $B_{\mu\nu}$. This *constant* parameter thus breaks ordinary Lorentz invariance.

On the other hand, the $\mathcal{N} = 1$ non(anti)commutative superspace has also been studied [5, 10]. This can be realized when there are R-R background (constant self-dual graviphoton background) in the superstring propagating space. In that case, non(anti)commutativity of $\mathcal{N} = 1$ superspace is realised on world sheet boundary

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \neq 0. \quad (2)$$

Here, $C^{\alpha\beta}$ is naively VEV of constant graviphoton field.

Quantum field theory on these non(anti)commutative (super)space has been studied in recent decade. For example, see the review [3, 4] and references therein for QFT on noncommutative space-time and recent work [6, 8, 9, 10, 11, 12] for QFT on non(anti)commutative superspace. There are many interesting properties in the framework of NC QFT *e.g.* UV/IR mixing, noncommutative instanton, relation to matrix model and so on. But almost all these works have been discussed in a formally Lorentz invariant approach and their representation corresponds to usual Poincaré algebra in spite of their violation of Lorentz symmetry.

But recently, deformed Lorentz invariance of this NC QFT was proposed [13, 14, 29] by using twisted Hopf algebra. By reinterpreting our fundamental symmetry as twisted deformed Hopf algebra, we can construct twisted Lorentz invariant quantum field theory on noncommutative space.

In this paper, we extend twisted deformed (Hopf) Poincaré algebra to twisted deformed *Poincaré SUSY* algebra and investigate whether this approach would be to what extent extensible. By choosing appropriate twist element $\mathcal{F} \in \mathcal{U}(\mathcal{SP}) \otimes \mathcal{U}(\mathcal{SP})$, we see twisted Lorentz invariant formulation of QFT on *non(anti)commutative superspace* is possible. And we also comment on the new noncommutativity of central charge coordinate.

The organization of this paper is as follows. Section 2 is a brief review of deformed Poincaré algebra especially twisted equation of P - P twist element and the work of Ref.[13]. Section 3 is extension to simple SUSY algebra and we see validity of our new fermionic twist element. Consistency between algebra and twisted symmetry is discussed in section 4. In section 5, we consider extended ($\mathcal{N} \geq 2$) SUSY algebra and establish some other kind of twist element. Then we claim new noncommutativity of central charge coordinate. Section 6 is our conclusion.

2 Twisted deformed (Hopf) Poincaré algebra

In this section, we review the twisted deformed Hopf Poincaré algebra following the recent work [13]. The review of the Hopf algebra itself is in Ref.[15, 16, 19].

The Poincaré algebra \mathcal{P} consists of Lorentz generators $M_{\mu\nu}$ and translation generators P_μ . This algebra contains abelian subalgebra P_μ . By using this subalgebra, we can construct twist element of quantum group. For more detail see Ref.[16]. This twist element permits to deform the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})$ (this is called trivial Hopf algebra that possesses quasitriangularity). There exist co-product $\Delta_0 : \mathcal{U}(\mathcal{P}) \longrightarrow \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$. For $X \in \mathcal{P}$, this is written as

$$\Delta_0(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X, \quad (3)$$

and our co-unit $\epsilon : \mathcal{U}(\mathcal{P}) \longrightarrow \mathcal{K}$ and antipode $\gamma : \mathcal{U}(\mathcal{P}) \longrightarrow \mathcal{U}(\mathcal{P})$ are

$$\begin{aligned} \epsilon(X) &= 0 \text{ (co unit),} \\ \gamma(X) &= -X \text{ (antipode),} \end{aligned} \quad (4)$$

for all $X \in \mathcal{P}$ and

$$\begin{aligned} \epsilon(\mathbf{1}) &= 1, \\ \gamma(\mathbf{1}) &= \mathbf{1}. \end{aligned} \quad (5)$$

\mathcal{K} is the base field of the vector space. After twisting the algebra by twist element \mathcal{F} , the co-product of twisted algebra $\mathcal{U}_t(P)$ is redefined by $\Delta_t(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}$. The consistency of Hopf algebra requires that after twisting, we have to change the definition of multiplication of original algebra representation [16, 17]

$$\begin{aligned} m(a \otimes b) \equiv ab \longrightarrow m_t(a \otimes b) &\equiv a \star b \\ &= m \circ \mathcal{F}^{-1}(a \otimes b). \end{aligned} \quad (6)$$

This formally can be considered as the origin of noncommutativity of space.

Twist element is constructed from abelian subalgebra P_μ [18]

$$\mathcal{F}^{PP} = \exp \left(\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu \right). \quad (7)$$

To be consistent with the property of Hopf algebra, the twist element \mathcal{F} should satisfy twist equation

$$\mathcal{F}_{12}(\Delta_0 \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}(\text{id} \otimes \Delta_0)(\mathcal{F}), \quad (8)$$

and co-unit condition

$$(\epsilon \otimes \text{id})(\mathcal{F}) = \mathbf{1} = (\text{id} \otimes \epsilon)(\mathcal{F}). \quad (9)$$

Notice that the co-product Δ_0 property admit

$$\begin{aligned}
(\text{id} \otimes \Delta_0) e^{X \otimes Y} &= (\text{id} \otimes \Delta_0) \sum_{n=0}^{\infty} \frac{1}{n!} (X \otimes Y)^n \\
&= (\text{id} \otimes \Delta_0) \sum_n \frac{1}{n!} X^n \otimes Y^n \\
&= \sum_n \frac{1}{n!} X^n \otimes \Delta_0(Y^n) \\
&= \sum_n \frac{1}{n!} X^n \otimes (\Delta_0(Y))^n \\
&= \sum_n \frac{1}{n!} (X \otimes \Delta_0(Y))^n = e^{X \otimes \Delta_0(Y)}
\end{aligned} \tag{10}$$

for bosonic quantities X, Y . By using this property and bosonic feature of generator P_μ , we can see this P - P twist element actually satisfy the twist equation as follows. The P - P twist element is the sum of the infinite series:

$$\begin{aligned}
\mathcal{F} &= \exp \left(\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \right)^n \Theta^{\mu_1 \nu_1} \dots \Theta^{\mu_n \nu_n} (P_{\mu_1} \otimes P_{\nu_1}) \dots (P_{\mu_n} \otimes P_{\nu_n}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \right)^n \Theta^{\mu_1 \nu_1} \dots \Theta^{\mu_n \nu_n} (P_{\mu_1} \dots P_{\mu_n}) \otimes (P_{\nu_1} \dots P_{\nu_n}) .
\end{aligned} \tag{11}$$

Co-product acts on the product P_μ s in such a way that

$$\begin{aligned}
\Delta_0(P_{\mu_1} \dots P_{\mu_n}) &= \prod_{i=1}^n (P_{\mu_i} \otimes \mathbf{1} + \mathbf{1} \otimes P_{\mu_i}) \\
&= \sum_{l=0}^n \binom{n}{l} P^l \otimes P^{n-l} ,
\end{aligned} \tag{12}$$

here P^l stands for the l times product of P_μ abstractly. Using above, and after appropriate reassignment of indices, we get

$$\begin{aligned}
\mathcal{F}_{12}(\Delta_0 \otimes \text{id})(\mathcal{F}) &= \exp \left(\frac{i}{2} \Theta^{\rho\sigma} P_\rho \otimes P_\sigma \right)_{12} (\Delta_0 \otimes \text{id}) \exp \left(\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^n \frac{1}{n! m!} \left(\frac{i}{2} \right)^{n+m} \binom{n}{l} \Theta^{\rho_1 \sigma_1} \dots \Theta^{\rho_m \sigma_m} \Theta^{\mu_1 \nu_1} \dots \Theta^{\mu_n \nu_n} \\
&\quad \times P_{\rho_1} \dots P_{\rho_m} P_{\mu_1} \dots P_{\mu_l} \otimes P_{\sigma_1} \dots P_{\sigma_m} P_{\mu_{l+1}} \dots P_{\mu_n} \otimes P_{\nu_1} \dots P_{\nu_n} ,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
& \mathcal{F}_{23}(\text{id} \otimes \Delta_0)(\mathcal{F}) \\
&= \sum_{n'=0}^{\infty} \sum_{m'=0}^{\infty} \sum_{l'=0}^{n'} \frac{1}{n'!m'!} \left(\frac{i}{2}\right)^{n'+m'} \binom{n'}{l'} \Theta^{\rho_1 \sigma_1} \dots \Theta^{\rho_{m'} \sigma_{m'}} \Theta^{\mu_1 \nu_1} \dots \Theta^{\mu_{n'} \nu_{n'}} \\
& \quad \times P_{\mu_1} \dots P_{\mu_{n'}} \otimes P_{\rho_1} \dots P_{\rho_{m'}} P_{\nu_1} \dots P_{\nu_{l'}} \otimes P_{\sigma_1} \dots P_{\sigma_{m'}} P_{\nu_{l'+1}} \dots P_{\nu_n} .
\end{aligned} \tag{14}$$

Each individual term in expansion can be characterized only by a set of three numbers; namely α , β and γ . α represents the number of $P_\mu P_\nu$ contraction through Θ between the first and second factors of the tensor product, β between the first and third factors, γ between the second and third factors. And we see the relation

$$\begin{cases} \alpha = m = l' \\ \beta = l = n' - l' \\ \gamma = n - l = m' \end{cases} ,$$

then n, m, l and n', m', l' are determined from these equations. If the term which relates to α , β , γ is found in n, m, l series, corresponding term always exist in n', m', l' expansion, and vice versa. Moreover these coefficients are identical,

$$\frac{1}{m!n!} \left(\frac{i}{2}\right)^{n+m} \binom{n}{l} = \frac{1}{m'!n'!} \left(\frac{i}{2}\right)^{n'+m'} \binom{n'}{l'} = \frac{1}{\alpha!\beta!\gamma!} \left(\frac{i}{2}\right)^{\alpha+\beta+\gamma} . \tag{15}$$

So the twist equation is satisfied order by order.

Co-unit condition is rather trivial because of the property

$$\epsilon(XY) = \epsilon(X)\epsilon(Y). \tag{16}$$

If we consider this P - P twist element and deform the original algebra, the multiplication m of original algebra representation has to be changed in the deformed algebra $\mathcal{U}_t(\mathcal{P})$,

$$m(x_\mu \otimes x_\nu) = x_\mu x_\nu \longrightarrow m_t(x_\mu \otimes x_\nu) = x_\mu \star x_\nu \tag{17}$$

This suggests noncommutativity of space-time coordinate. Indeed, because translation generator acts on coordinate as $P_\mu x_\nu = i\eta_{\mu\nu}$, we find

$$\begin{aligned}
m_t(x_\mu \otimes x_\nu) &= x_\mu \star x_\nu = m \circ e^{-\frac{i}{2}\Theta^{\alpha\beta}P_\alpha \otimes P_\beta}(x_\mu \otimes x_\nu) \\
&= m \circ \left[x_\mu \otimes x_\nu + \frac{i}{2}\Theta^{\alpha\beta}\eta_{\alpha\mu} \otimes \eta_{\beta\nu} \right] \\
&= x_\mu x_\nu + \frac{i}{2}\Theta_{\mu\nu},
\end{aligned} \tag{18}$$

then this gives noncommutative relation among space-time coordinates

$$[x_\mu, x_\nu]_\star = i\Theta_{\mu\nu} \neq 0. \tag{19}$$

This exactly correspond to Moyal-Weyl star product bracket.¹

¹In practice, we are considering space-space noncommutativity because time direction noncommutativity may alarm the causality and unitarity problem.

3 Construction of twisted Poincaré SUSY algebra

It is not difficult to extend ordinary Hopf algebra to super (\mathbf{Z}_2 -graded) Hopf algebra [20]. The definition of Hopf algebra basically contains graded algebra. So, we can include fermionic generator, *i.e.* supercharge Q_α . The key point is how to construct the twist element $\mathcal{F} \in \mathcal{U}(\mathcal{SP}) \otimes \mathcal{U}(\mathcal{SP})$. Here $\mathcal{U}(\mathcal{SP})$ is universal enveloping of Poincaré SUSY algebra. We first consider the simplest $\mathcal{N} = 1$ SUSY algebra

$$\begin{aligned} [P_\mu, Q_\alpha] &= 0, \quad [M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \\ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu, \quad \{Q_\alpha, Q_\beta\} = 0. \end{aligned} \quad (20)$$

We here omit $\bar{Q}_{\dot{\alpha}}$ sector. Because Hopf algebra is defined for this graded algebra, we can choose abelian subsector as P_μ, Q_α sector. Choosing these generators lead to new twist element in $\mathcal{U}(\mathcal{SP}) \otimes \mathcal{U}(\mathcal{SP})$, for example

$$\mathcal{F}^{QQ} = \exp \left[-\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta \right], \quad (21)$$

$C^{\alpha\beta}$ in the equation is some constant. This twist element satisfies the Yang-Baxter equation

$$\mathcal{F}_{12} \mathcal{F}_{13} \mathcal{F}_{23} = \mathcal{F}_{23} \mathcal{F}_{13} \mathcal{F}_{12}. \quad (22)$$

Because supercharge Q_α is grassmann odd, the exponential does end at finite order

$$\begin{aligned} \mathcal{F} &= \exp \left[-\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta \right] \\ &= \mathbf{1} \otimes \mathbf{1} - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta - \frac{1}{8} C^{\alpha\beta} C^{\alpha'\beta'} Q_\alpha Q_{\alpha'} \otimes Q_\beta Q_{\beta'}. \end{aligned} \quad (23)$$

It is not obvious that this element satisfy the twist equation because of their non-trivial sign that arise when interchanging the two grassmann odd quantities. But this finite expansion of exponent enable us to check the validity of this Q - Q twist by straightforward calculation. Let us check whether this twist element satisfy the twist equation (8). The left hand side of Eq. (8) is

$$\begin{aligned} \text{LHS} &= \mathcal{F}_{12} (\Delta_0 \otimes \text{id})(\mathcal{F}) \\ &= \left(\mathbf{1} \otimes \mathbf{1} - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta - \frac{1}{8} C^{\alpha\beta} C^{\gamma\delta} Q_\alpha Q_\gamma \otimes Q_\beta Q_\delta \right)_{12} \\ &\quad \times \left[\Delta_0(\mathbf{1}) \otimes \mathbf{1} - \frac{1}{2} C^{\kappa\tau} \Delta_0(Q_\kappa) \otimes Q_\tau - \frac{1}{8} C^{\kappa\tau} C^{\lambda\pi} \Delta_0(Q_\kappa Q_\lambda) \otimes Q_\tau Q_\pi \right] \\ &= \left(\mathbf{1} \otimes \mathbf{1} - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta - \frac{1}{8} C^{\alpha\beta} C^{\gamma\delta} Q_\alpha Q_\gamma \otimes Q_\beta Q_\delta \right)_{12} \\ &\quad \times \left[\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{1}{2} C^{\kappa\tau} (Q_\kappa \otimes \mathbf{1} \otimes Q_\tau + \mathbf{1} \otimes Q_\kappa \otimes Q_\tau) \right. \\ &\quad \left. - \frac{1}{8} C^{\kappa\tau} C^{\lambda\pi} (Q_\kappa Q_\lambda \otimes \mathbf{1} \otimes Q_\tau Q_\pi + Q_\kappa \otimes Q_\lambda \otimes Q_\tau Q_\pi \right. \\ &\quad \left. - Q_\lambda \otimes Q_\kappa \otimes Q_\tau Q_\pi + \mathbf{1} \otimes Q_\kappa Q_\lambda \otimes Q_\tau Q_\pi) \right]. \end{aligned} \quad (24)$$

and similar for RHS. We used here the graded tensor product property

$$(a \otimes b)(a' \otimes b') = (-)^{|b||a'|} (aa' \otimes bb'), \quad (25)$$

where $|a|$ is fermion number of a , and also co-product property

$$\begin{aligned} \Delta_0(XY) &= \Delta_0(X)\Delta_0(Y) \\ &= (X \otimes \mathbf{1} + \mathbf{1} \otimes X)(Y \otimes \mathbf{1} + \mathbf{1} \otimes Y) \\ &= XY \otimes \mathbf{1} + X \otimes Y + (-)^{|X||Y|} Y \otimes X + \mathbf{1} \otimes YX. \end{aligned} \quad (26)$$

for $X, Y \in \mathcal{SP}$. Be careful, after expanding these quantities explicitly, for higher order $\mathcal{O}(C^3)$ term that involve three supercharge in one tensor sector gives vanishing contribution. Then, it is not difficult to confirm that left hand side is precisely equal to right hand side. So the twist element quantity Eq. (21) is valid. This twist leads to non(anti)commutativity of $\mathcal{N} = 1$ superspace coordinate. Actually, because supercharge acts on fermionic coordinate θ^α as $Q_\alpha \theta^\beta = i\delta_\alpha^\beta$, we see

$$\begin{aligned} m_t(\theta^\alpha \otimes \theta^\beta) &= \theta^\alpha \star \theta^\beta \\ &= m \circ e^{\frac{1}{2}C^{\gamma\delta}Q_\gamma \otimes Q_\delta}(\theta^\alpha \otimes \theta^\beta) \\ &= m \circ \left[\theta^\alpha \otimes \theta^\beta + \frac{1}{2}C^{\gamma\delta}\delta_\gamma^\alpha \otimes \delta_\delta^\beta \right] \\ &= \theta^\alpha \theta^\beta + \frac{1}{2}C^{\alpha\beta}. \end{aligned} \quad (27)$$

This gives

$$\{\theta^\alpha, \theta^\beta\}_\star = C^{\alpha\beta} \neq 0. \quad (28)$$

Simultaneously, additional effects occur on other commutation relations,

$$\begin{aligned} [x^\mu, x^\nu]_\star &= C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu \sigma_{\beta\dot{\delta}}^\nu \bar{\theta}^{\dot{\gamma}} \bar{\theta}^{\dot{\delta}}, \\ [x^\mu, \theta^\alpha]_\star &= -iC^{\alpha\beta} \sigma_{\beta\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}}, \end{aligned} \quad (29)$$

it may be eliminated if one takes chiral coordinates.

This corresponds to Q -deformation (non-supersymmetric deformation) of non(anti)commutative supersymmetric theory [28]².

Note, however, we can choose only chiral part *or* anti-chiral part of supercharge because there exist non-trivial anticommutator $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} \neq 0$. This allows chiral part noncommutativity $\{\theta_\alpha, \theta_\beta\} \neq 0$ or anti-chiral part noncommutativity $\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} \neq 0$ but not both. Actually, this is possible only when we consider Euclidean (4+0) or Atiyah-Ward (2+2) space-times [5, 22].

Other interesting twist are possible. For example

$$\mathcal{F}^{PQ} = \exp \left[\frac{i}{2} \lambda^{\mu\alpha} (P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu) \right] \quad (30)$$

²But in twisted SUSY case, half of SUSY does not broken because our original algebra is intact after twisting.

gives mixed noncommutativity between bosonic and fermionic coordinates

$$\begin{aligned}[x^\mu, x^\nu]_\star &= \lambda^{\mu\alpha} \sigma_{\alpha\dot{\beta}}^\nu \bar{\theta}^{\dot{\beta}} - \lambda^{\nu\alpha} \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}}, \\ [x^\mu, \theta^\alpha]_\star &= i\lambda^{\mu\alpha} \neq 0, \\ \{\theta^\alpha, \theta^\beta\}_\star &= 0.\end{aligned}\tag{31}$$

It should be confirmed too that this twist element indeed satisfy twist equation. Unlike the ordinary noncommutative case Eq. (7) (pure bosonic case), it is less obvious whether this fermionic twist element satisfies the twist equation and Yang-Baxter equation because of bothersome sign flips from interchanging grassmann odd quantity. But regarding $\lambda^{\mu\alpha} Q_\alpha$ as *bosonic* quantity, the same procedure in preceding section can be applied to the twist equation.

We can consider more general setting such as

$$\mathcal{F} = \exp \left[\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu + \frac{i}{2} \lambda^{\mu\alpha} (P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu) - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta \right], \tag{32}$$

where $\Theta^{\mu\nu}, C^{\alpha\beta}$ are some grassmann even constants and $\lambda^{\mu\alpha}$ odd one. The fact that the all generators P_μ, Q_α (anti)commute with each other allows one to calculate the twist equation more easily. In fact, all $\Theta^{\mu\nu} P_\mu \otimes P_\nu$, $\lambda^{\mu\alpha} P_\mu \otimes Q_\alpha$, $\lambda^{\mu\alpha} Q_\alpha \otimes P_\mu$, $C^{\alpha\beta} Q_\alpha \otimes Q_\beta$ are commute, then exponential will be factorized

$$\mathcal{F} = \mathcal{F}^{PP} \mathcal{F}^{QQ} \mathcal{F}^{PQ}. \tag{33}$$

Notice that if each of this factorized sector satisfy the twsit equation separately, then we can see that all combined exponential Eq. (32) also satisfy the twist equation. For example, if P - P twist and Q - Q twist satisfy the twist equation

$$(\mathcal{F}^{PP})_{12} (\Delta_0 \otimes \text{id}) \mathcal{F}^{PP} = (\mathcal{F}^{PP})_{23} (\text{id} \otimes \Delta_0) \mathcal{F}^{PP}, \tag{34}$$

$$(\mathcal{F}^{QQ})_{12} (\Delta_0 \otimes \text{id}) \mathcal{F}^{QQ} = (\mathcal{F}^{QQ})_{23} (\text{id} \otimes \Delta_0) \mathcal{F}^{QQ} \tag{35}$$

then

$$\begin{aligned}& \left(e^{\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta} \right)_{12} (\Delta_0 \otimes \text{id}) \left(e^{\frac{i}{2} \Theta^{\mu\nu} P_\mu \otimes P_\nu - \frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta} \right) \\ &= (\mathcal{F}^{PP})_{12} (\mathcal{F}^{QQ})_{12} \left(e^{\frac{i}{2} \Theta^{\mu\nu} \Delta_0(P_\mu) \otimes P_\nu - \frac{1}{2} C^{\alpha\beta} \Delta_0(Q_\alpha) \otimes Q_\beta} \right) \\ &= (\mathcal{F}^{PP})_{12} \left(e^{\frac{i}{2} \Theta^{\mu\alpha} \Delta_0(P_\mu) \otimes P_\nu} \right) (\mathcal{F}^{QQ})_{12} \left(e^{-\frac{1}{2} C^{\alpha\beta} \Delta_0(Q_\alpha) \otimes Q_\beta} \right) \\ &= (\mathcal{F}^{PP})_{12} (\Delta_0 \otimes \text{id}) \mathcal{F}^{PP} (\mathcal{F}^{QQ})_{12} (\Delta_0 \otimes \text{id}) \mathcal{F}^{QQ}.\end{aligned}\tag{36}$$

After using Eq. (34) and Eq. (35) and factorization condition, we find this (P - P + Q - Q) twist element actually satisfy the twist equaton. This is valid for P - Q sector too and Yang-Baxter equation is easier to prove.

This twist element allows genelal noncommutative superspace relation

$$\begin{aligned}[x^\mu, x^\nu]_\star &= i\Theta^{\mu\nu} + C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu \sigma_{\beta\dot{\delta}}^\nu \bar{\theta}^{\dot{\gamma}} \bar{\theta}^{\dot{\delta}} + \lambda^{\mu\alpha} \sigma_{\alpha\dot{\beta}}^\nu \bar{\theta}^{\dot{\beta}} - \lambda^{\nu\alpha} \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}}, \\ [x^\mu, \theta^\alpha]_\star &= i\lambda^{\mu\alpha} - iC^{\alpha\beta} \sigma_{\beta\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}}, \\ \{\theta^\alpha, \theta^\beta\}_\star &= C^{\alpha\beta},\end{aligned}\tag{37}$$

and any other commutator involving anti-chiral part of fermionic coordinates is zero.

Another type of non(anti)commutativity of superspace (called κ -deformation) is considered in Ref.[30].

4 Consistency

Let us check the transformation rule of noncommutative parameters. These show the consistency of transformation property and algebra. Because merely the co-product of Lorentz generator $M_{\mu\nu}$ and anti-supercharge $\bar{Q}^{\dot{\alpha}}$ are deformed by twisting, it is sufficient to show what these transform the non(anti)commutative parameters into.

4.1 P - P sector

For the P - P twisted sector, the transformation property of noncommutative parameter $\Theta^{\mu\nu}$ was already shown in Ref.[13]. In this case, the twisted co-product of Lorentz generator $M_{\mu\nu}$ is

$$\begin{aligned} \Delta_t^{PP}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &\quad - \frac{1}{2} \Theta^{\rho\sigma} [(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu} - \eta_{\sigma\nu} P_\mu)]. \end{aligned} \quad (38)$$

The co-product of $\bar{Q}^{\dot{\alpha}}$ in this stage is not changed. Then the action of twisted Lorentz generator to the twisted function $f_{\rho\sigma}^t \equiv x_{\{\rho} \star x_{\sigma\}}^3$ is $M_{\mu\nu}^t f_{\rho\sigma}^t = m_t \circ (\Delta_t((M_{\mu\nu})(x_{\{\rho} \otimes x_{\sigma\})))$.

We take $x^{[\rho} \otimes x^{\sigma]}$ to see the noncommutative parameter $\Theta^{\rho\sigma}$ and find

$$m_t \circ \left(\frac{1}{i} \Delta_t^{PP}(M_{\mu\nu}) (x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho) \right) = M_{\mu\nu}^t(\Theta^{\rho\sigma}) = 0. \quad (39)$$

Although the left-hand side of this equation looks like as tensor, it transforms in the twisted Lorentz invariant way, *i.e.* constant.

4.2 Q - Q sector

For Q - Q twisted sector, our co-product of Lorentz generator is

$$\begin{aligned} \Delta_t^{QQ}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &\quad + \frac{i}{2} (\sigma_{\mu\nu})_\alpha^\gamma C^{\alpha\beta} (Q_\beta \otimes Q_\gamma + Q_\gamma \otimes Q_\beta), \end{aligned} \quad (40)$$

and anti-supercharge is

$$\begin{aligned} \Delta_t^{QQ}(\bar{Q}^{\dot{\alpha}}) &= \bar{Q}^{\dot{\alpha}} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{Q}^{\dot{\alpha}} \\ &\quad + C^{\gamma\delta} \epsilon^{\dot{\alpha}\dot{\beta}} \{(\sigma^\rho)_{\gamma\dot{\beta}} P_\rho \otimes Q_\delta - Q_\gamma \otimes (\sigma^\rho)_{\delta\dot{\beta}} P_\rho\}. \end{aligned} \quad (41)$$

³This corresponds to the function $f^{\rho\sigma} = x^\rho x^\sigma$ in the commutative space. Detailed explanation is written in Ref.[13].

Then, twisted actions on the function $h^{\alpha\beta} = \theta^\alpha \theta^\beta$ are

$$m_t \circ \left(\Delta_t^{QQ}(M_{\mu\nu}) (\theta^\alpha \otimes \theta^\beta) \right) = -i (\sigma_{\mu\nu})_\gamma^\alpha \theta^\gamma \theta^\beta - i (\sigma_{\mu\nu})_\gamma^\beta \theta^\alpha \theta^\gamma, \quad (42)$$

$$m_t \circ \left(\Delta_t^{QQ}(\bar{Q}^{\dot{\alpha}}) (\theta^\alpha \otimes \theta^\beta) \right) = 0. \quad (43)$$

Twisted Lorentz transformation of non(anti)commutative parameter $C^{\alpha\beta}$ can be calculated

$$m_t \circ \left(\Delta_t^{QQ}(M_{\mu\nu}) (\theta^\alpha \otimes \theta^\beta + \theta^\beta \otimes \theta^\alpha) \right) = M_{\mu\nu}^t(C^{\alpha\beta}) = 0, \quad (44)$$

$$(\bar{Q}^{\dot{\alpha}})^t(C^{\alpha\beta}) = 0. \quad (45)$$

So, non(anti)commutative parameter $C^{\alpha\beta}$ is also twisted Lorentz invariant.

4.3 P - Q sector

The co-products in this case are

$$\begin{aligned} \Delta_t^{PQ}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &\quad - \frac{1}{2} \lambda^{\rho\alpha} [(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \otimes Q_\alpha - Q_\alpha \otimes (\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \\ &\quad - P_\rho \otimes (\sigma_{\mu\nu})_\alpha^\gamma Q_\gamma + (\sigma_{\mu\nu})_\alpha^\gamma Q_\gamma \otimes P_\rho], \end{aligned} \quad (46)$$

$$\begin{aligned} \Delta_t^{PQ}(\bar{Q}^{\dot{\alpha}}) &= \bar{Q}^{\dot{\alpha}} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{Q}^{\dot{\alpha}} \\ &\quad + \lambda^{\kappa\gamma} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\rho)_{\gamma\dot{\beta}} (P_\kappa \otimes P_\rho - P_\rho \otimes P_\kappa). \end{aligned} \quad (47)$$

So, same calculation shows

$$m_t \circ \left(\frac{1}{i} \Delta_t(M_{\mu\nu}) (x^\rho \otimes \theta^\alpha - \theta^\alpha \otimes x^\rho) \right) = M_{\mu\nu}^t(\lambda^{\rho\alpha}) = 0, \quad (48)$$

$$\bar{Q}^{\dot{\alpha}t}(\lambda^{\rho\alpha}) = 0. \quad (49)$$

Then $\lambda^{\mu\alpha}$ is twisted Lorentz invariant.

4.4 Mixed sector

Actually, in P - P + Q - Q + P - Q mixed twist sector it results in only a linear combination of the three sectors:

$$\begin{aligned}
\Delta_t^{Mix}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\
&\quad - \frac{1}{2} \Theta^{\rho\sigma} [(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu} P_\nu - \eta_{\sigma\nu} P_\mu)] \\
&\quad + \frac{i}{2} (\sigma_{\mu\nu})_\alpha{}^\gamma C^{\alpha\beta} (Q_\beta \otimes Q_\gamma + Q_\gamma \otimes Q_\beta) \\
&\quad - \frac{1}{2} \lambda^{\rho\alpha} [(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \otimes Q_\alpha - Q_\alpha \otimes (\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu)] \\
&\quad - [P_\rho \otimes (\sigma_{\mu\nu})_\alpha{}^\gamma Q_\gamma + (\sigma_{\mu\nu})_\alpha{}^\gamma Q_\gamma \otimes P_\rho], \tag{50}
\end{aligned}$$

$$\begin{aligned}
\Delta_t^{Mix}(\bar{Q}^{\dot{\alpha}}) &= \bar{Q}^{\dot{\alpha}} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{Q}^{\dot{\alpha}} \\
&\quad + C^{\gamma\delta} \epsilon^{\dot{\alpha}\dot{\beta}} \{(\sigma^\rho)_{\gamma\dot{\beta}} P_\rho \otimes Q_\delta - Q_\gamma \otimes (\sigma^\rho)_{\delta\dot{\beta}} P_\rho\} \\
&\quad + \lambda^{\kappa\gamma} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\rho)_{\gamma\dot{\beta}} (P_\kappa \otimes P_\rho - P_\rho \otimes P_\kappa). \tag{51}
\end{aligned}$$

All (anti)commutation relations are transformed as following.

$$\begin{aligned}
m_t \circ (\Delta_t(M_{\mu\nu})(x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho)) &= i [\lambda^{\sigma\alpha} (\sigma^\rho)_{\alpha\dot{\alpha}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}} - \lambda^{\rho\alpha} (\sigma^\sigma)_{\alpha\dot{\alpha}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\gamma}}] \bar{\theta}^{\dot{\gamma}}, \\
m_t \circ (\Delta_t(M_{\mu\nu})(x^\rho \otimes \theta^\alpha - \theta^\alpha \otimes x^\rho)) &= C^{\gamma\alpha} (\sigma_\rho)_{\gamma\dot{\kappa}} (\bar{\sigma}_{\mu\nu})^{\dot{\kappa}}{}_{\dot{\gamma}} \bar{\theta}^{\dot{\gamma}}, \\
m_t \circ (\Delta_t(M_{\mu\nu})(\theta^\alpha \otimes \theta^\beta + \theta^\beta \otimes \theta^\alpha)) &= 0, \tag{52}
\end{aligned}$$

$$\begin{aligned}
m_t \circ (\Delta_t(\bar{Q}^{\dot{\alpha}})(x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho)) &= -i C^{\gamma\delta} \epsilon^{\dot{\alpha}\dot{\beta}} \left[(\sigma^\rho)_{\gamma\dot{\beta}} (\sigma^\sigma)_{\delta\dot{\delta}} \bar{\theta}^{\dot{\delta}} - (\sigma^\rho)_{\gamma\dot{\gamma}} \bar{\theta}^{\dot{\gamma}} (\sigma^\sigma)_{\delta\dot{\beta}} \right] \\
&\quad - i \epsilon^{\dot{\alpha}\dot{\beta}} [\lambda^{\rho\gamma} (\sigma^\sigma)_{\gamma\dot{\beta}} - \lambda^{\sigma\gamma} (\sigma^\rho)_{\gamma\dot{\beta}}], \\
m_t \circ (\Delta_t(\bar{Q}^{\dot{\alpha}})(x^\rho \otimes \theta^\alpha - \theta^\alpha \otimes x^\rho)) &= -C^{\gamma\alpha} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\rho)_{\gamma\dot{\beta}}, \\
m_t \circ (\Delta_t(\bar{Q}^{\dot{\alpha}})(\theta^\alpha \otimes \theta^\beta + \theta^\beta \otimes \theta^\alpha)) &= 0. \tag{53}
\end{aligned}$$

These consist with the transformation of Eq. (37), such that non(anti)commutative parameters are invariant, while coordinates are transformed precisely.

5 Extended non(anti)commutative (super)space

Non(anti)commutativity of extended superspace $\{\theta_\alpha^I, \theta_\beta^J\} = C_{\alpha\beta}^{IJ} \neq 0$ has also been considered. Especially $\mathcal{N} = 2$ non(anti)commutative superspace [21] and field theory on it was studied [22, 7, 23, 24]. But extended Poincaré SUSY algebra

$$\begin{aligned}
[P_\mu, Q_\alpha^I] &= 0, \quad [M_{\mu\nu}, Q_\alpha^I] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^I, \\
\{Q_\alpha^I, \bar{Q}_\beta^J\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta^{IJ}, \quad \{Q_\alpha^I, Q_\beta^J\} = \varepsilon_{\alpha\beta} Z^{IJ}, \tag{54}
\end{aligned}$$

shows that it looks impossible to use Q_α^I generator to construct twist element because of their non-abelianity⁴. Instead, we can use central charge generator Z^{IJ} because central charge always commutes with any other generators. Now we set up

$$\mathcal{F} = \exp \left(\frac{i}{2} \Xi_{IJ} Z^I \otimes Z^J \right), \quad (55)$$

Ξ_{IJ} is some constant. This Z - Z twist also satisfies the twist equation. The proof is the same as P - P twist by their bosonic abelian property.

Central charge coordinate formulation of supersymmetric theory [25, 26, 27] is needed when there are nonzero central charge in the algebra. Central charge generator $Z^I = \frac{\partial}{\partial z_I}$ act on the central charge coordinate z_J as

$$Z^I z_J = \delta_J^I \quad (56)$$

then, one can easily compute the commutator of this coordinate

$$\begin{aligned} m_t(z_I \otimes z_J) &= z_I \star z_J = m \circ e^{-\frac{i}{2} \Xi_{KL} Z^K \otimes Z^L} (z_I \otimes z_J) \\ &= m \circ \left[z_I \otimes z_J + \frac{i}{2} \Xi_{KL} \delta_I^K \otimes \delta_J^L \right] \\ &= z_I z_J + \frac{i}{2} \Xi_{IJ}. \end{aligned} \quad (57)$$

Now Ξ_{IJ} is antisymmetric with respect to I, J , we obtain

$$[z_I, z_J]_\star = i \Xi_{IJ} \neq 0. \quad (58)$$

This is noncommutative relation among central charge coordinate. But this situation is possible only when we consider extended $\mathcal{N} \geq 3$ SUSY. The reason is that $\mathcal{N} = 1$ SUSY does not contain central charge and $\mathcal{N} = 2$ SUSY have only one central charge $Z^{12} = -Z^{21} \equiv Z$ so it always commute.

6 Conclusion

In this paper, we extend twisted deformed (Hopf) Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ based on Ref.[13] to supersymmetric one $\mathcal{U}_t(\mathcal{SP})$. Considering this symmetry as our fundamental symmetry, we can set up quantum field theory on non(anti)commutative space by twisted Lorentz invariant way. More practical work was done in Ref.[14] for bosonic (non supersymmetric) sector. This approach may useful for recent developing supersymmetric gauge theory on non(anti)commutative superspace. We hope these reinterpretation of symmetry algebra helps us to consider quantum field theory on noncommutative space correctly and ameliorates some ambiguity of representation. In our approach, it seems that $\mathcal{N}=1$ twisted SUSY remain completely unbroken, because our twisting procedure doesn't change the original algebra. Which situation is different from

⁴But in some special setting, for example $\mathcal{N} = 2$ singlet deformed superspace can admit Lorentz invariant deformation of superspace. For more detail, see Ref.[21]

ordinary approach that breaks half of SUSY and settle down to $\mathcal{N} = 1/2$ SUSY after turning on non(anti) commutativity of superspace [5]. It may give a new physical insight on this reinterpreted formalism. In this paper, we consider only $\mathcal{N} = 1$ simplest non(anti)commutative space. It seems difficult to extend to $\mathcal{N} \geq 2$ case. But instead we can consider the theory on noncommutative central charge coordinate z_I . It is interesting to investigate the physical meaning of this noncommutativity. These subjects are the future research interest that relate this twisted algebraic approach.

A Appendix

Following Ref.[13], our super Poincaré algebraic relation and representation of this algebra is *not* deformed from ordinary case but product should be replaced by star product (twisted multiplication). So, in case of commutative limit, we can easily recover the usual representation and product between functions.

A.1 Super Poincaré algebra

Our notation of super Poincaré algebra is as follows.

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho}, \\
[M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu, \\
[P_\mu, Q_\alpha^I] &= 0, \quad [P_\mu, \bar{Q}_{\dot{\alpha}}^I] = 0, \\
[M_{\mu\nu}, Q_\alpha^I] &= i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta^I, \quad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}, \\
\{Q_\alpha^I, \bar{Q}_{\dot{\beta}}^J\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta^{IJ}, \\
\{Q_\alpha^I, Q_\beta^J\} &= \varepsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}_{\dot{\alpha}}^I, \bar{Q}_{\dot{\beta}}^J\} = \varepsilon_{\dot{\alpha}\dot{\beta}} Z_{IJ}^*.
\end{aligned} \tag{59}$$

Translation generator P_μ , Lorentz generator $M_{\mu\nu}$ and supercharge Q_α in $\mathcal{N} = 1$ super Poincaré algebra can be represented as differential operator on superspace

$$\begin{aligned}
P_\mu &= i\partial_\mu, \\
M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) - i\theta^\alpha(\sigma_{\mu\nu})_\alpha^\beta \frac{\partial}{\partial\theta^\beta} - i\bar{\theta}_{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}, \\
Q_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma_{\beta\dot{\alpha}}^\mu\partial_\mu.
\end{aligned} \tag{60}$$

We take the convention that $\sigma_{\mu\nu} = -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$.

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